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TWO GEOMETRICAL EXAMPLES FROM ARISTOTLE'S *METAPHYSICS*

εὑρίσκεται δὲ καὶ τὰ διαγράμματα ἐνεργείαι· διαιροῦντες γὰρ εὑρίσκουσιν. εἰ δ' ἦν διηιρημένα, φανερὰ ἂν ἦν· νῦν δ' ἐνυπάρχει δυνάμει. διὰ τί δύο ὀρθαὶ τὸ τρίγωνον; ὅτι αἱ περὶ μίαν στιγμὴν γωνίαι ἴσαι δύο ὀρθαῖς. εἰ οὖν ἀνῆκτο ἡ παρὰ τὴν πλευράν, ἰδόντι ἂν ἦν εὐθὺς δῆλον διὰ τί. ἐν ἡμικυκλίωι ὀρθὴ καθόλου διὰ τί; ἐὰν ἴσαι τρεῖς, ἥ τε βάσις δύο καὶ ἡ ἐκ μέσου ἐπισταθεῖσα ὀρθή, ἰδόντι δῆλον τῶι ἐκεῖνο εἰδότι. ὥστε φανερὸν ὅτι τὰ δυνάμει ὄντα εἰς ἐνέργειαν ἀγόμενα εὑρίσκεται· αἴτιον δὲ ὅτι ἡ νόησις ἐνέργεια· ὥστ' ἐξ ἐνεργείας ἡ δύναμις, καὶ διὰ τοῦτο ποιοῦντες γιγνώσκουσιν (ὕστερον γὰρ γενέσει ἡ ἐνέργεια ἡ κατ' ἀριθμόν).

Aristotle, *Metaphysics Θ* 9. 1051 a 21–33

27 διὰ τί EJA^b Al. (ap. Ross et Heath): διότι Al^p (ap. Jaeger) Lat. recc. post τί (26, 27) interpunxerunt Cannan et Ross, sed fortasse Al. post δῆλον (26) et τί (27) post δῆλον (26) et καθόλου (27) interpunxerunt qui διότι scripserunt (e.g. Bekker, Christ, Jaeger) 30 ἡ νόησις ci. Ross: νόησις ἡ codd.

The discussion of mathematical knowledge and its relation to the construction of an appropriate diagram in Aristotle's *Metaphysics Θ* 9. 1051 a 21–33 is an important, if compressed, account of Aristotle's most mature thoughts on mathematical knowledge. The discussion of what sort of previous knowledge one must have for understanding a theorem recalls the discussion at *An. Post. A* 1. 71 a 17–21, where the epistemological point is similar and the examples the same. The first example, that the interior angles of a triangle equal two right angles, appears no less than thirty times in the corpus (inter alia, *An. Post A* 5. 74 a 16, 23. 84 b 6–9, 24. 86 a 22–30). The example of the angle inscribed in a semicircle being a right angle also occurs at *An. Post. B* 11. 94 a 27–34, but in a very different context from its two companions. Illustrations of both theorems provided clear stock examples for Aristotle.

It is not a simple matter, however, to decide what the diagrams and the proofs for the respective theorems must have been. Yet the philosophical and mathematical points seem to rely on the way we understand the constructions. In this paper I shall first argue that Aristotle's proof of the angles-in-the-triangle theorem was the so-called Pythagorean version,[1] and not the Euclidean, as Heiberg, Heath, and Ross claim.[2] Then I offer an emendation of the text to bring the second example into conformity with the philosophical point expressed in the text.

Aristotle's goal in this passage is to show how the constructions make knowledge possible. The ekthesis being complete, all one has to do is see the appropriate relations and recall the right theorems for the proof and the theorem to be evident. He offers

[1] Proclus, *in Primum Euclidis*, p. 379. 2–5, attributes this ascription to Eudemus, the Peripatetic. The fact that Aristotle's friend is the source of this claim itself suggests Aristotle's acquaintance with this version.

[2] Heiberg, pp. 19–20, followed by Heath, 1949, pp. 29–30, 217, and Ross, 1924, ii. 269–70. Cf. Heath, 1926, i. 321. For Bibliographical details see pp. 371–2.

two examples where one does not see the proof until one has all the constructions laid out which are needed for the proof. Any reconstruction of his diagram or proof must take into account the fact that all one is given is the initial figure determined by the statement of the theorem, a triangle or an angle in a semicircle. Any construction to be made must be stipulated only then. And then the proof must be evident, given the theorem appealed to. Other theorems will, of course, be needed. But the example is pointless if it dispenses with the cited theorem or relegates it to a minor role.[3]

Euclid I. 32 has two parts: I. 32a, in every triangle when one side is extended the outside angle is equal to the two opposite interior angles, and I. 32b, the three interior angles of a triangle are equal to two right angles. The given figure in the first part will not be a triangle, but a triangle with one side extended. The geometer then constructs a parallel CE to AB (Fig. 1) in order to prove I. 32a. I. 32b then follows trivially from the construction. But observe that to prove I. 32b no new constructions are necessary. So in this example the geometer starts with more than a triangle and his figure ends with more than a triangle and a parallel line.

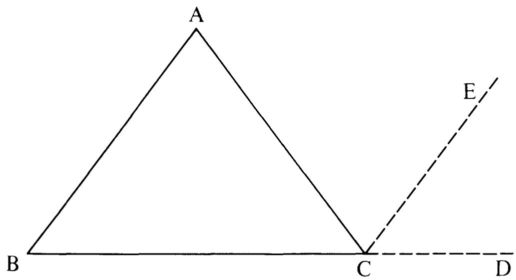

Figure 1

The Pythagorean proof involves merely the construction of a parallel line. The theorem is immediately evident from the given figure, the construction and the obvious facts about alternate angles in cut parallel lines, and the fact that DAB + BAC + CAE is two right angles (Fig. 2). Aristotle says (*Met.* Θ 9. 1051a24–26), 'Why is the triangle two right angles? Because the angles about a point are equal to two right angles. So if the parallel to a side were drawn it would be immediately evident to someone who sees (the figure) why it is.'[4] Aristotle starts with a triangle and tells you to construct a parallel line. He cannot have the Euclidean proof in mind. For one cannot assume that one has CD already constructed from his remark, unless he already assumes

[3] In this discussion I assume *τὰ διαγράμματα* at 1051a22 means diagrams (Ross, 1924, ii. 268), and not geometrical proofs of propositions (as in Bonitz, 1849, p. 407), unless the proofs are seen as the constructed figures. Heath (1949, p. 216) supposes it to mean any geometrical proofs or propositions. So it can sometimes, but not always (cf. Bonitz, 1870, p. 178, lines 3–11), and certainly not here, where the discussion is about what propositions must be known and, more important, what geometrical construction is needed. Division here is not the division of the *Topics* or Plato's *Sophist*. It is geometrical analysis, as Bonitz (1849, p. 407) indicates. We must understand the *διαγράμματα* as being found (1051a22), divided (a23), and led into actuality (a29–30). For we must understand the objects to be divided (*διαιροῦντες*) to be of the same sort as that which get divided, namely *τὰ διαγράμματα*, which must also be *τὰ δυνάμει ὄντα* (1051a29). And as Bonitz (1848, pp. 407–8) says, these are certainly mathematical objects. For a list of parallel passages where there is the point that from the diagram the matter is evident, cf. Heiberg, p. 6.

[4] *διὰ τί δύο ὀρθαὶ τὸ τρίγωνον; ὅτι αἱ περὶ μίαν στιγμὴν γωνίαι ἴσαι δύο ὀρθαῖς. εἰ οὖν ἀνῆκτο ἡ παρὰ τὴν πλευράν, ἰδόντι ἂν ἦν εὐθὺς δῆλον διὰ τί.*

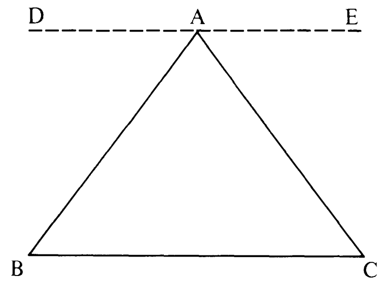

Figure 2

I. 32a. But then the parallel line is no longer needed. It has been drawn. Furthermore Aristotle does not even mention I. 32a anywhere in the corpus. Even the theorem which gives the total exterior angles of a polygon (*An. Post. A* 24. 85b37–86a3, *B* 17. 99a19) relies on I. 32b.[5]

That a single construction is required is clear from the traditional interpretation. Both ps.-Alexander and Eustathius, the Arabic translator, take ἀνῆκτο ἡ παρὰ τὴν πλευράν to mean 'the line along the side is extended'.[6] Thus they begin with a triangle, extend one side, and then apply I. 32a, which they suppose to have been proved. There is one construction, and, moreover, the passage now reflects even the wording of Euclid. For once I. 32a is proved, namely that ACD = CAB + ABC, all that one needs is the observation that the angles about the point C are two right angles. And this is just about the only significant fact that Euclid brings in. If the author of this part of ps.-Alexander's commentary is really Michael of Ephesus, and if Eustathius translated in the mid-ninth century,[7] then we may suppose this interpretation to have been nigh standard thence till Schwegler, who considers the phrase ambiguous.[8] Indeed the interpretation is so elegant that it is to be preferred, if it is possible.

Bonitz, however, drew attention to *Topics Θ* 3. 158b31, where ἡ παρὰ τὴν πλευράν can only refer to the parallel to the side. This would not clinch the matter (Aristotle could have used the phrase in both ways) were it not for the preponderance of evidence that the expression cannot mean 'the line along the side'.[9] And so we are forced to reject the interpretation of ps.-Alexander and Eustathius.

[5] Heath suggests (1921, pp. 144, 340; 1949, p. 63) that this theorem is Pythagorean. This is enticing, but he offers no evidence. When Proclus mentions the theorem (pp. 383–4) he claims that it, with the general theorem of the interior angles of a polygon, is a preliminary for the *Timaeus*. Even if he has special reason for this, it will not push the theorem into the world of the Pythagoreans.

[6] Ps.-Alexander, p. 586. 12–15; Averroes, vi. 1214. 1–2.

[7] Cf. Bouyges' remarks in Averroes v. cxviii–cxix.

[8] Cf. Schwegler, p. 185. Aquinas, *In Met.* ix. L. 10: C 1889, adopts ps.-Alexander's interpretation, and even those who still put in both constructions, e.g. Averroes, vi. 1216. 6–1217. 1, and Niphius, p. 509, take the construction to which Aristotle refers to be the extended line, CD. Albertus Magnus apparently is unclear as to which construction he adopts (cf. Bürke, p. 144). I gather that the less ambiguous, but obviously corrupt ἡ περὶ τὴν πλευράν was accepted because it better describes the accepted construction. Hence the texts of Erasmus or Petrus de la Rovière and the common translation with 'qui circa latus est', which may also be found, for whatever reason, in the Parma Aquinas (p. 547).

[9] It could in the appropriate context mean 'the area applied to a side'. Cf. Mugler's article. Knorr, pp. 196–7, has given further evidence for the expression's indicating a parallel in Archimedes, although it is rare in Euclid. If his overall argument is correct it certainly points to pre-Euclidean usage.

What gives rise to this interpretation further counts against Heiberg, Heath, and Ross. For the interpretation can only have arisen from people who wanted to reconcile the text with the familiar proof in Euclid. But they saw that there had to be one construction. And the Heiberg interpretation gives two. So they interpreted the passage in a way that would give one construction and a proof.[10]

Heiberg, Heath and Ross[11] offer one objection to the occurrence here of the Pythagorean proof. Heath claims that ἀνῆκτο must refer to an ordinate, and he cites Apollonius, where ἀνάγειν typically is said of a line drawn from points on a diameter to points on a circumference. κατάγειν typically takes lines from a circumference to a diameter. The suggestion is that the lines cannot be drawn parallel to the base. Of course Figure 2 could easily be drawn so that the line DE is drawn up and down parallel to BA. Moreover it is not clear what relevance Apollonian terminology has here.[12] In the only other geometrical use of ἀνάγειν in Aristotle (*Meteor. Γ* 5. 376a 1) αἱ ἀπὸ τοῦ ηκ ἀναγόμεναι γραμμαί are drawn from a diameter to the circumference. But from the vantage of the man looking at the diagram all directions are possible.[13] In our case there is no circumference or diameter, and in the technical use these are the only direction-markers. Unless we assume that the direction is understood as hidden in the construction of the parallel, the technical use cannot be used as a guide to whether Aristotle in *Θ* 9 means by ἀνῆκτο that one literally draws up from a given line. Nor will Eudemus on the Pythagorean version and Euclid's *Elements* adjudicate the matter. They both use ἤχθω. But fortunately Euclid's *Data* does, i.e. if one accepts definitions 13, 14:[14]

> *ιγ'. Κατηγμένη ἐστὶν ἡ ἀπὸ δεδομένου σημείου ἐπὶ θέσει εὐθεῖαν ἀγομένη εὐθεῖα ἐν δεδομένηι γωνίαι.*
> The line led from a given point to a line in position in a given angle is 'led down'.
>
> *ιδ'. Ἀνηγμένη ἐστὶν ἡ ἀπὸ δεδομένου σημείου πρὸς θέσει εὐθείαι ἀγομένη εὐθεῖα ἐν δεδομένηι γωνίαι.*
> The line led from a given point on a line in position in a given angle is 'led up'.

In the context of analysis, given a point in position on a line and an angle, ἀνάγειν is to draw a line at that point in that angle. Direction is not relevant, but the angle is. Given the uniqueness of our passage and this evidence of neutrality from Euclid (or Apollonius) we are free to translate it by the equally idiomatic 'is led back (from A)'. Finally I should point out that Heath himself is not so confident of his argument. For he says, 'This (Aristotle's description of the construction) would apply equally to the Pythagorean way of drawing the parallel or to Euclid's'.[15]

[10] Cf. Bürke, p. 133, who recognizes this as a clear problem.

[11] See note 2.

[12] Cf. Apollonius 1. 50, 51, where ἀνάγειν does not imply direction up.

[13] This is not the appropriate place to show this. But one locus of lines will lead to the horizon, i.e. behind HK.

[14] It is really of no importance if they do belong to Euclid or to Apollonius, as the tradition referred to by the scholium to definitions 13–15 maintains. For if an Apollonian definition fits this view of ἀνάγειν then there is no reason to reject the same usage for Aristotle.

[15] Heath, 1949, p. 73. The variations which one may conjure up are perhaps endless. But the number of cases which fit the text are not. So we propose a version with one construction, CE parallel to AB as in Fig. 1. But then we will have to use something like the fact that the interior angles on the same side of two cut parallels equal two right angles. And this version must be rejected, not because it lacks ancient authority, but because we will not then use the fact that the angles about a point are two right angles, even if this fact is used in proof of the parallel theorem (as Euclid I. 28, 29).

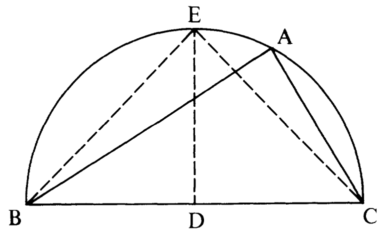

Figure 3

Unfortunately, as the text stands, the second example of the angle in the semicircle in *Met.* *Θ* 9 is both philosophically and mathematically incorrect. The text says that we are to see why the angle in the semicircle is *universally* a right angle. Yet the construction we are supposed to make seems to yield only the particular case where the inscribed triangle is isosceles (cf. Fig. 3). Even if the diagram in Figure 3 can yield an immediate proof (without new constructions), the example would be poor. It is not immediately evident that the universal case is true from the diagram. Are we supposed to see why the theorem is true on the basis of an incomplete induction, an anathema to generality which at best would give the appearance of being a proof by cases? If this account is correct, either Aristotle has made a foolish mistake, or we shall have to emend the text.

Before setting ourselves the onerous task of emendation we should consider the one avenue of avoiding the dilemma. Heiberg,[16] followed by Heath[17] and Ross,[18] points out that Aristotle may have in mind an appeal to Euclid III. 21: in a circle the angles in the same segment are equal to one another. So using this theorem the problem will be solved in the following manner. According to most of the standard editions (Bekker, Ross, Jaeger), the text (1051a27–29) should read, 'If three sides are equal, the base taken as two and the side brought up upright from the middle [D], [why or that the angle in the semicircle is universally right] is evident to someone who knows this [that the interior angles of a triangle equal two right angles] upon seeing [Fig. 3].'

Construct DE perpendicular to BC and connect EB, EC. Now BEC is clearly right. For BD = DC = ED. And so EBD + BED + BDE = two right angles. So EBD + BED = one right angle. But EBD = BED. So BED = one half of a right angle. Likewise DEC = one half of a right angle. So BEC = BED + DEC = one right angle. This proof is thus far cumbersome because it must use two facts, the angles-of-a-triangle theorem and that ED is perpendicular to BC. Now having shown this, Aristotle can appeal to Euclid III. 21 and say that BAC is a right angle as well.

It is not an objection to this version that the proof is unlike Euclid's proof. In this we shall see that Heath is correct. However, there is good reason to reject this solution. First, it would be nice to have some textual justification for appealing to III. 21 other than that it makes the particular case general.[19] III. 21 occurs nowhere in the

[16] Heiberg, p. 21.

[17] Heath, 1921, pp. 339–40; 1926, ii. 63–4; 1949, pp. 73–4.

[18] Ross, 1924, ii. 271; 1949, p. 641.

[19] In Eudemus' account of the squaring of lunules by Hippocrates of Chios, he refers to an analogue of III. 21 and III. 31 (cf. Simplicius, *in Phys*, vol. 1, p. 61. 14–18): *καὶ γωνίας ἴσας δέχεται τὰ ὅμοια τμήματα. αἱ γοῦν τῶν ἡμικυκλίων πάντων ὀρθαί εἰσι, καὶ ⟨αἱ⟩ τῶν μειζόνων ἐλάττονες ὀρθῶν καὶ τοσούτωι ὅσωι μείζονα ἡμικυκλίων τὰ τμήματα, καὶ αἱ τῶν ἐλαττόνων μείζονες καὶ τοσούτωι ὅσωι ἐλάττονα τὰ τμήματα.* This seems to indicate that the theorem of the angle in the semicircle was proved separately from the case where the segments

Aristotelian corpus; so it might seem strange that Aristotle appeals to I. 32b when the important theorem is III. 21. The natural rejoinder to this is that in the first example Aristotle has appealed to a simple fact, when there are more 'advanced' facts used in the proof, i.e. about parallel lines. So under any account the fact that the angles about a point are two right angles and I. 32b must be brought in for a special reason. Later I shall give such an account. Suffice it for now that Heath's solution leaves it in the air why Aristotle mentions I. 32b and not III. 21.

A second difficulty with Heath's account is that we have too many constructions. Under the hypothesis that we begin by 'dividing' a figure, that figure must be the angle in the semicircle BAC. So when we construct ED, we have yet to construct BE, EC. But the text tells us to construct one line, not three. Of course we could have any initial triangle given by the angle constructed, but only luck would make it the right one for the altitude to be drawn from the centre. So Ross (1949, p. 644) draws the one isosceles triangle in this manner. I think, however, that a natural reading of *An. Post.* *A* 1 suggests a proof with one angle in the semicircle which is initially given.[20] If it is possible to find such a proof, we should adopt it for the later passage at *B* 11 on which Ross is commenting.

This leads to the third objection to Heath's example. The example at *An. Post.* *A* 1 is clearly the same as that at *Met.* *Θ* 9. But only by a contortion could one have the proof end as suggested by the text, *An. Post.* *B* 11. 94a28–34:

Why is the angle in the semicircle right? Upon what being the case is it right? Let A stand for right angle, B for half two right angles, and C for the angle in the semicircle. Then B is the cause of A, the right angle, belonging to C, the angle in the semicircle. For this (B) is equal to A (cf. Figure 4), and C is equal to B. For B is half two right angles. So since B is half two right angles, A belongs to C (and this is that the angle in the semicircle is right).[21]

The conclusion on the basis of Heath's version of the proof would have to be that the particular angle in the semicircle is right, and then by III. 21 that they all are. But this is not what Aristotle says. The difficulty of moulding this passage into a part of Heath's proof is particularly evident from Ross's attempt to do so.[22] For Ross gets to this deduction only by sacrificing any mention of the fact that a perpendicular has been joined.[23] It will be seen that the wording of Ross's proof minus the perpendicular must be correct. For as Ross correctly sees, the three passages should have the same proofs at hand, whatever they may be.

Heath, on the other hand, sees the difficulty which Ross gets into and adopts the

of the circle are greater and less. Moreover the separate cases of the segments were divided into two parts, one giving the relation of the angle size to the right angle, and the second giving its equivalence in terms of its 'distance' from the right angle.

[20] *An. Post.* *A* 1. 71a19–21: *ὅτι μὲν γὰρ πᾶν τρίγωνον ἔχει δυσὶν ὀρθαῖς ἴσας, προῄδει· ὅτι δὲ τόδε τὸ ἐν τῶι ἡμικυκλίωι τρίγωνόν ἐστιν, ἅμα ἐπαγόμενος ἐγνώρισεν.* The proof starts with I. 32b and the triangle constructed by the angle in the semicircle as givens. The proof then should use both facts.

[21] *διὰ τί ὀρθὴ ἡ ἐν ἡμικυκλίωι; τίνος ὄντος ὀρθή; ἔστω δὴ ὀρθὴ ἐφ' ἧς Α, ἡμίσεια δυοῖν ὀρθαῖν ἐφ' ἧς Β, ἡ ἐν ἡμικυκλίωι ἐφ' ἧς Γ. τοῦ δὴ τὸ Α τὴν ὀρθὴν ὑπάρχειν τῶι Γ τῆι ἐν τῶι ἡμικυκλίωι αἴτιον τὸ Β. αὕτη μὲν γὰρ τῆι Α ἴση, ἡ δὲ τὸ Γ τῆι Β· δύο γὰρ ὀρθῶν ἡμίσεια. τοῦ Β οὖν ὄντος ἡμίσεος δύο ὀρθῶν τὸ Α τῶι Γ ὑπάρχει (τοῦτο δ' ἦν τὸ ἐν ἡμικυκλίωι ὀρθὴν εἶναι).*

[22] Ross, 1949, p. 691; also cf. Ross, 1924, ii. 271.

[23] Ross, 1924, ii. 271, thinks the right angle a superfluous oversight: 'it would be natural enough for Aristotle by an oversight to think that the angle could more easily be proved to be right in the symmetrical case in which it is the sum of two half right angles'. This is the sort of error which would be very much unnatural to the author of the discussion on universal proof at *An. Post.* *A* 5.

alternative proof in Euclid, which uses I. 32a.[24] This isolates the example in *An. Post.* *B* 11 from its twin at *A* 1. That he needs to assume two different proofs are being alluded to here and at *Met.* *Θ* 9 is a further thorn in his argument. However, Heath sees that otherwise he will have to assume that the angle in the semicircle is the specific one in Figure 3. No more here than at *Met* *Θ* 9 is there a hint of a need for generalization.

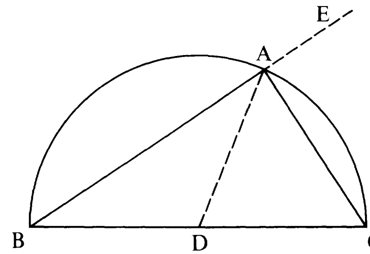

Figure 4

In defence of his reading Heath argues that ps.-Alexander seems to think that there are two problems, the special case and the application of III. 21. In fact ps.-Alexander's own text may have divided the question into two parts. For he seems to punctuate his text,[25] ἰδόντι ἂν ἦν εὐθὺς δῆλον διὰ τί ἐν ἡμικυκλίωι ὀρθή; καθόλου διὰ τί; for *Met.* *Θ* 9. 1051a26, 27. Apart from substituting σταθεῖσα for ἐπισταθεῖσα, he reads lines 28–29 canonically. He takes the subject of each δῆλον (lines 26, 28) as ὅτι, i.e. what is made evident is the fact of the conclusion. Ps.-Alexander clearly does not understand the example. He explains the example of the angle in the semicircle by citing Euclid's proof, but adapted to or confused with the canonical reading of line 28, i.e. he gives Euclid's proof, but only for the special case of the isosceles triangle. He insists on AD (Figure 4) being perpendicular to BC, ἤχθω δὲ πρὸς ὀρθῆς τῆι βγ (BC) ἡ αε (AD).[26] Although he offered some insight on the first example, as a guide to the sense of this text, ps.-Alexander is not very helpful.

However, whoever punctuated the copy of the *Metaphysics* used by ps.-Alexander understood the problem and so put the two questions in the text, the first of which ps.-Alexander answers inappropriately with a proof adapted from Euclid. But this editor must have assumed Aristotle to have left the second question, of the universality of the theorem, completely unanswered. Heath ingeniously adopts the sense of the passage which results from this punctuation. But the revised punctuation is not a good reason for Heath's position. It is merely a statement of the problem which Heath tries to solve. Whoever adopted the punctuation adopted it just because he could not see the second question answered. But if there are two questions at least one, the first, will be answered in the text. Heath manages to answer both questions, thereby showing *his* ingenuity as geometer and ps.-Alexander's lack of ingenuity. But as to the idea conveyed, ps.-Alexander was still closer to the point being made. Ps.-Alexander botches the example, and Heath makes it too complex and perplexing. Whatever figure Aristotle intended, it should involve no more than one construction, and the theorem should be evident from the figure and Theorem I. 32b.

Before proceeding to the solution, one should see how the proof in Euclid of the

[24] Heath, 1949, p. 72.
[25] Ps.-Alexander, p. 596, lines 16–20, 23–4.
[26] Ibid., p. 596. 34.

angle-in-the-semicircle theorem (Euclid III. 31) fits with Aristotle's texts. Given triangle ABC in semicircle ABDC, extend BA to E and connect AD. Because triangles ABD, ACD are isosceles, BAD = ABD, ACD = DAC. But BAD + DAC = BAC. Hence ABD + ACD = BAC. But by I. 32a, ACD + ABD = EAC. So EAC = BAC, so that each is a right angle. This proof fits with *An. Post. B* 11. So Heath could have chosen it as easily as the alternative proof. On two counts, however, it fails our conditions for being an appropriate version. It has a superfluous construction AE, and it uses I. 32a instead of I. 32b. But it is interesting to note that given his position on the first example in *Met. Θ* 9, Ross could have accepted ps.-Alexander's version, or some other adaption of Euclid's proof.

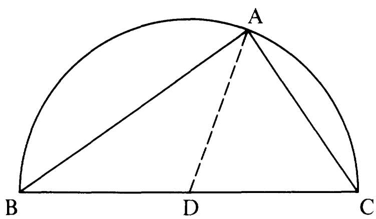

Figure 5

I believe that from a correct reading of *Met. Θ* 9 we can reconstruct Aristotle's construction and proof. It will be seen that contrary to what Heath argued,[27] it is possible to get all three passages, *An. Post. A* 1, *B* 11 and *Met. Θ* 9, in harmony. This involves severing *ὀρθή* from *ἡ ἐκ μεσοῦ ἐπισταθεῖσα*. Then the antecedent of the sentence at 1051 b27–29 will be, ⟨*διότι*⟩ *ἐὰν ἴσαι τρεῖς, ἥ τε βάσις δύο καὶ ἡ ἐκ μεσοῦ ἐπισταθεῖσα*. How we are to do this leads us to the appropriate emendation. The point for now is to see the sense which the result of such an emendation would bring. We take Ross's proof for the *Analytics* [Ross, 1949, p. 641] and use the figure for the alternative proof in Euclid. That proof is inappropriate because it uses I. 32a. For it shows BAC = ABD + ACD and the sum of all three equal to the angle about point D via I. 32a (cf. Figure 5). The current proof uses I. 32b, at the climax of the proof, and then may be thought to end with the reasoning at *An. Post. B* 11. Finally it uses only the one construction demanded by the emended text. The text may be translated, 'If three sides are equal, the base taken as two and the side standing from the middle (to the apex),...'

So given semicircle BAC and triangle ABC, join AD where D is the centre of the circle. Following the reasoning of the three passages in Aristotle on the theorem, we observe that ABC is a triangle and hence by I. 32b that ABC + BAC + ACB = two right angles (*An. Post. B* 11). As before BD = AD = DC; the initial conditions are then met. Now DBA = BAD and DAC = ACD. Hence BAC = BAD + DAC = DBA + DCA. So BAC + BAC = two right angles. From here the reasoning of *An. Post. B* 11 may proceed. Let X stand for 'right angle', Y for 'half two right angles', and Z for 'angle in the semicircle', i.e. for BAC. We have shown that Y belongs to (is equal to) Z. And X is equal to Y. So X is equal to Z.[28]

It is also clear from the example why Aristotle singles out for mention the claims,

[27] Heath, 1949, pp. 72–3.

[28] This version of the proof is proffered and rejected by Ross, 1924, ii. 271. So far as I know, Dancy, pp. 374–5, is the only scholar to have endorsed it for *An. Post. B* 11. He does not say how he deals with *Met. Θ* 9.

in the first example, that the angle about a point is two right angles and, in the second, that the angles in the triangle are two right angles. First there is the obvious matter of the middle term as providing understanding, to which I shall return. But almost as important is that these two geometrical facts are the last to occur in the respective proofs. Afterwards the mind uses matters more general than geometry. In other words Aristotle means by 'it is clear to one who knows the fact that...' that one knows the various relevant theorems and geometrical facts, this and this and this last one. This is why the examples are so natural, and why Heath's introduction of III. 21 is not.

The difficulty is to make the text conform with the necessary interpretation. Simplest is to excise *ὀρθή* from the text.[29] Ross, who chose not to use the Latin-Arabic text in his edition of the *Metaphysics*,[30] rightly says that all the evidence available to him counts against deleting *ὀρθή*. However, if the author of this part of ps.-Alexander is Michael of Ephesus, the Arabic translation of Eustathius will be the oldest witness to the text, even if a bad one.[31] The Arabic text does omit *ὀρθή*, but at some cost. For the translation is very free (omitting some punctuation): 'and likewise why in every semi-circle is there one right angle on the circumference whether its (the triangle's) sides are equal or are unequal if its base is the diameter whether they fall on the centre of the circumference or elsewhere, then it will be clear to someone who looks at it and has knowledge of it.'[32] This is one instance where the Arabic text seems too interpretative to base a conclusive argument on. Yet deletion of *ὀρθή* may be the best solution.

It is easy to offer a host of other plausible emendations which preserve *ὀρθή*. The problem is simply to move the comma to get something like Christ's *καθόλου; διότι ἐὰν ᾗ τε...ἐπισταθεῖσα, ὀρθή· ἰδόντι...*[33] But Christ loses the parallel between line 26, where the apodosis is a *δῆλον* clause, and line 28, where the apodosis was a *δῆλον*

[29] This, in effect, is what Roman does in his translation of Aquinas. However, he offers no justification for it, and it would seem there is none in any of the Latin translations from the Greek. Cf. Aquinas, p. 694. The only justification he can give for his translation is that Aquinas gives the alternative proof in Euclid, without assuming the line drawn to be perpendicular. But his translation, which is supposed to represent Aquinas's text, is not here based on any edition I know of. The Latin of the Arabic translation used by Averroes, which I shall presently discuss, is a reasonable paraphrase and does not resemble Roman's translation (cf. Bürke, p. 68). Aquinas's discussion shows the influence of Averroes. Ironically, the proof Aquinas gives in his commentary on the *An. Post.* (XVIII, p. 200), which uses the construction of the perpendicular, fits *Met.* *Θ* 9, just as the version given here better fits that passage, as if he had forgotten which goes with which.

[30] Ross, 1924, i. clxiv.

[31] Walzer claims that the Arabic text represents an independent tradition.

[32] Averroes, p. 1214, lns. 4–6. The Arabic goes 'wa kadhaalika lima fii kulli niṣfi daa'iratin qaa'imatun waahidatun ?alaa 'lmahiiti tasaawit 'a suuquhaa 'am lam tatasaawi 'idhaa kaana qaa?idatuhaa 'lquṭra ?alaa wuṣti 'lmaḥiiṭi waqa?at 'am ?alaa gayrihi fa'innahu bayyana liman nadḥara 'ilayhu bi'inna lahu ma?rifatu dhaalika'. I think the attempt to reconstruct the Greek text is in vain, the first part of the protasis up to *ἐκ μέσου* being very free with the text. The Arabic is bizarre also because it gives no construction at all. It just says that the theorem is obvious.

[33] Aristotle, 1886. Christ's recommendation appears in his notes. By removing *ὀρθή* from the protasis, we are not committed to keeping the figure which we would get with *ὀρθή*. I would like to think this is what Christ intended. If so he has the right construction, but a bad text. Heath (1949, p. 73) notes that he translated from Christ's edition in the first edition of his text of Euclid (1926, first ed. 1908, ii. 63), though he does not adopt Christ's suggested punctuation. He there translates the relevant phrase, 'the third set up at right angles at its middle point'. Like Eustathius, Heath takes the middle to be the middle of the circumference. In the later work he adopts Ross's interpretation. Except for his preferring *διὰ τί* to Christ's *διότι* in line 27, his change of heart is over the interpretation, not the text.

clause before emendation. So argues Ross, and he seems right.[34] Also Christ's suggestion will make the cause a result of the particular construction rather than a universal geometrical fact. In the previous example, the cause was that the angles about a point are equal to two right angles, and not that the angles about the apex on the constructed parallel are equal to two right angles. One would expect Aristotle, on Christ's suggestions, to have said in the second example that the radii of circles are equal, or some such universal fact.[35]

Nor will the obvious ἐπισταθεῖσα, ὀρθὴ ἰδόντι δῆλον... work. For impersonal δῆλον should have a conjunction, such as ὅτι or διὰ τί, governing the subordinate clause.[36]

Cannan also tries to remove ὀρθή from the protasis by writing διὰ τί for ἰδόντι and supposing that the apodosis is ὀρθὴ διὰ τί; or as Ross punctuates ὀρθὴ διὰ τί. δῆλον... Again Cannan loses the symmetry of apodoses. Ross also argues that the corruption is unlikely.[37] But with the punctuation unsure at lines 26–27, in order to get either Ross's text or ps.-Alexander's it is possible that one of the several διὰ τί was lost in transmission. The text would then have perfect parallelism with line 26 in the manuscripts:

..., ἰδόντι ἂν ἦν εὐθὺς δῆλον διὰ τί. ἐν ἡμικυκλίωι ὀρθὴ καθόλου διὰ τί; ἐὰν ἴσαι τρεῖς, ἥ τε βάσις δύο καὶ ἡ ἐκ μέσου ἐπισταθεῖσα, διὰ τί ὀρθὴ ἰδόντι δῆλον τῶι ἐκεῖνο εἰδότι.

Moreover, there are several ways the corruption could have come about.

It is possible, with ps.-Alexander and ps.-Philoponus,[38] to understand a that-clause as the subject for the second δῆλον, at line 28, though not for the first at line 26. The

[34] Ross, 1924, ii. 271.

[35] The plethora of διὰ τί in Ross's text displeases Jaeger, although they are supported by the main Greek manuscripts. Jaeger argues in favour of the Latin translation, also favoured by Bekker and Christ, which puts διότι for διὰ τί after καθόλου; in line 27, and so makes διὰ τί in line 26 go with the following clause. Besides the weak manuscript tradition behind it, Ross had objected that this reading requires a translation such as 'Why is the angle in a semicircle always a right angle? Because, if..., it is clear to someone who...that...' The fact that the theorem is evident becomes the explanation of its truth. Jaeger's defence involves the ways of referring to middle expressions at line 24 and *An. Post. B* 94a37, 95a14, to show that the reply to the 'why' question is a 'because' answer, whether διότι or ὅτι, followed perhaps by the expression of the middle expression. Of course the middle expression cannot be the construction. The middle expression which διότι would initiate is referred to later by ἐκεῖνο. One would have a sentence which could be filled out, '(The angles of a triangle equal two right angles) because, if a certain construction is made, it will be clear to someone who sees the figure and knows that fact'. The structure is, in fact, very close to the previous argument. But if Jaeger is right, to prevent the properties of the auxiliary constructions from becoming the cause of the theorem, ὀρθή must be emended to something like δύο ὀρθαῖς, with the result that we would have a strong hyperbaton. Note that if the hyperbaton is possible here, it is possible without emendation for Ross's text. Given that the alternative reading is better supported, and makes sense, I see no reason to adopt the Vulgate reading. Hence we might prefer one of two other plausible emendations based on the text as it appears in Ross's edition.

[36] The only case in Aristotle attested (and doubted) by Bonitz (1870, p. 173b38–40) is *Politica E* II. 1313a18; Vahlen emends δῆλον to δῆλον ὅτι, and Ross to δηλονότι.

[37] Ibid. In his capacity as Secretary to the Delegates of the Oxford University Press, C. Cannan made recommendations to Ross and allowed him to use some unpublished emendations, of which, I presume, the emendation under discussion is one. Cf. ibid. i. v.

[38] Ps.-Philoponus on the *Metaphysics* may have read ὅτι ὀρθή. At least that is what he understands as what is clear: ἅμα τῶι θεάσασθαι τινὰ τὴν ἐν τῶι ἡμικυκλίωι γωνίαν, δήλη ἂν ἦν αὐτῶι ὅτι ὀρθή ἐστιν (cod. urb. 49, p. 110v). Unfortunately the author does not seem aware that constructions may be necessary or even what the problem is (cf. Appendix for how Philoponus fails to understand the example).

lost parallelism could be explained by the fact that, for Aristotle, something is most evident to us only when we know why. *ὅτι ὀρθή* could easily have become *ὀρθῇ*, especially as soon as an editor associated *ὀρθή* with *ἐπισταθεῖσα*. Alternatively *δηλονότι τῶι* could have become *δῆλον τῶι* by simple haplography. Either will do. There is then reason for thinking the original text to have been:

. . . , ἰδόντι ἂν ἦν εὐθὺς δῆλον διὰ τί. ἐν ἡμικυκλίωι ὀρθὴ καθόλου διὰ τί; ἐὰν ἴσαι τρεῖς, ἥ τε βάσις δύο καὶ ἡ ἐκ μέσου ἐπισταθεῖσα, ὅτι ὀρθὴ ἰδόντι δῆλον τῶι ἐκεῖνο εἰδότι.

For there is a strong parallel between this reading and those passages where Aristotle is making roughly similar points about the diagram making the theorem evident. In particular *Meteor.* *Γ* 5. 375b17–19 provides a parallel.[39] From the diagram the fact that the rainbow is not greater than a semi-circle is evident. This is similar to what Aristotle has been saying in our passage in the *Metaphysics*. For to make a fact clear is to give an explanatory account.

All this, however, is a matter of taste. Though I prefer deletion of *ὀρθή*, I am not certain which of the suggested emendations is correct. But I am certain that one of these or one like these must be correct. And if I am correct, these two examples from *Met.* *Θ* 9 provide, from the philosophical point of view, the finest examples of what Aristotle feels important in geometrical analysis to be found in the corpus. When thus cleansed of their obscurity, they show clearly the relation that Aristotle wants to emphasize between the construction of a figure and the process of acquiring geometrical understanding through a middle term.

They also point to another interesting historical possibility. We have several theorems, all not Euclidean (I am including the exterior-angle theorem) and all somewhat interrelated. In one case the proof is alleged by Aristotle's friend Eudemus to be Pythagorean. His source can only derive from the 'Pythagoreans' in the Academy. Another is traditionally associated with the *Timaeus*. Neither of the examples from *Met.* *Θ* 9 uses Theorem I. 32a, while their cousins in Euclid do. It is not implausible to suppose that Euclid introduced I. 32a and proceeded to use it, even where it was traditionally unnecessary. For the proof in Euclid of III. 31 is certainly less elegant than the one I claim to find in Aristotle. At the very least this is further indication that the state even of Book I of the *Elements* was not so well established as we might have thought.

We may also ask why Aristotle is at all interested in I. 31b and III. 31. I. 31b is certainly a model of a simple and fine example. But I would suggest that there is another reason why these two examples appear together. For Aristotle the major open question is whether the circle can be squared. So for example at *An. Priora* *B* 25. 69a30–36 he considers whether the circle can be squared by lunes. For lunes can be made equal to rectangles and rectangles squared. Elsewhere (*de Anima* *B* 2. 413a16–20), he says that we have the explanation of squaring when we see that a rectangle is squared by constructing a mean proportional. But the construction of the mean proportional in Euclid VI. 13 uses III. 31. Again it is a not implausible hypothesis that Aristotle is interested in III. 31 because it is employed in a construction of the mean proportional. But the use of mean proportion is not found in the proof that the rectangle can be squared, Euclid II. 14, the crowning proof of that book. For

[39] *ὅτι δ' οὔτε κύκλον οἷόν τε γίγνεσθαι τῆς ἴριδος οὔτε μεῖζον ἡμικυκλίωι τμῆμα, καὶ περὶ τῶν ἄλλων τῶν συμβαινόντων περὶ αὐτήν, ἐκ τοῦ διαγράμματος ἔσται θεωροῦσι δῆλον.* Another less likely possible emendation is based on a parallel at *Meteor.* *A* 8. 345a22: *δῆλος ἡμῖν ἅπας ὁ κύκλος*. But cf. note 37, where ps.-Philoponus has something like this.

II. 14 properly includes the construction of VI. 13, and in Aristotle's terminology obscures the middle term, i.e. in squaring the rectangle it makes no mention of the implicitly constructed mean proportion. III. 31 is, then, a significant buttress of the theory of proportions which was coming into existence at the time *An. Post.* was written and which receives such important notice in *An. Post. A* 5. Thus we see both that III. 31 plays a role in the new theory of proportion and that it is part of the progression of theorems aimed at squaring the circle. So too, according to Eudemus of Rhodes, Hippocrates used III. 31 as a starting point in the squaring of lunes. From the theorem of the equality of alternating angles and the fact that the angles about a point are two right angles we get the theorem of the interior angles of a triangle. From that and the fact that the two angles of an isosceles triangle are equal (*An. Priora A* 24. 41 b 14–22) we get the fact that the angle in a semicircle is right. From that we get the construction of the mean proportion in Euclid, VI. 13, and then the squaring of every rectangle. And this is the first step to squaring the circle. Taken together the structure and interrelation of these theorems offer a glimpse of pre-Euclidean introductory geometry as taught and developed by the Italians in the Academy.

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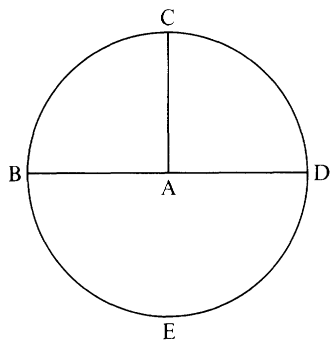

Figure 6

APPENDIX

It is curious, if not bizarre, that every Greek commentator on Aristotle who has anything substantial to say on the matter of III. 31 constructs the perpendicular to the diameter. Ps-Alexander's case is understandable, and he could be right and I wrong. He is merely adapting Euclid to the text of Aristotle which he has. The other two, Philoponus, *in Arist. Post. An.*, p. 377, followed by Eustratius, *in Arist. Post. An. B*, p. 141, are different matters. They clearly do not understand what Aristotle's example is. They construct a circle BCDE with centre at A, and then draw perpendicular CA to drawn diameter BD (cf. Fig. 6). So far we have something like Aristotle's text of the *Metaphysics*, but apparently no angle in the semicircle. Then, however, they argue that, since a line drawn to a line makes two right angles, angles BAC and DAC are right angles. The reasoning is spurious, since the CA was drawn perpendicular, and invalid. But the interesting point is that Philoponus and Eustratius think that BAC and DAC are angles in the semi-circle. That what else they say harmonizes with *An. Post. B* 11 is small solace.

Philoponus rightly identifies III. 31 as occurring in the third book of Euclid. But it is quite clear that he never read that book, nor was able to read it. So when he goes on to say that the theorem is difficult (!) to show to the ungeometrical from what Euclid says, he means himself, or at least ought to.[40] How did our error creep into *Met.* Θ 9? It seems to me that someone such as Philoponus with his misconception emended Θ 9 so as to conform with his idea of what the angle in the semicircle is supposed to be. Naturally, this meant that the perpendicular had to be constructed, as that is the only 'angle in the semicircle' which will be right. The error would have to have occurred in an era of extreme mathematical ignorance, e.g. the sixth or seventh century, and in a place where there was no Simplicius or Proclus.[41]

BIBLIOGRAPHY

Alexander Aphrodisiensis, *In Aristotelis Metaphysica Commentaria*, ed. M. Hayduck, Berlin, 1891.

Aquinas, Thomas, *Opera Omnia*, ed. Petrus Fiaccadori, Parma, 1852–73 (citations from 2nd. printing).

—, *In Aristotelis Analyticorum Duos Libros*, vol. xviii of above, 1865.

—, *In Aristotelis Metaphysicorum* ..., vol. xx of above, 1866.

—, *In Aristotelis Metaphysica*, 2 vols., trans. J. P. Roman, Chicago, 1961.

Aristotle, *Opera Aristotelis*, vol. ii, ed. Erasmus, Basel, 1531.

—, *Operum Aristotelis*, vol. ii, ed. Petrus de la Rovière, Geneva, 1606.

—, *Opera*, 5 vols., ed. I. Bekker, Berlin, 1831–1870.

—, *Metaphysica*, ed. W. Christ, Leipzig, 1886.

—, *Metaphysica*, ed. W. Jaeger, Oxford, 1957.

—, *Politica*, ed. W. D. Ross, Oxford, 1957.

Averroes, *Tafsir ma ba'd at-Tabia't ('Grand Commentaire' de la Métaphysique)*, ed. M. Bouyges, in *Bibliotheca Arabica Scholasticorum*, vols. v–vii, 1938–1952.

Bonitz, H., *Aristotelis Metaphysica*, Bonn, 1849.

—, *Index Aristotelicus*, vol. v of Bekker's Aristotle, 1870.

Bürke, B., *Das Neunte (Θ) Buch des Lateinischen Grossen Metaphysik-Kommentars von Averroes*, Bern, 1969.

Dancy, R., 'On Some of Aristotle's Second Thoughts about Substances: Matter', *Philosophical Review*, 87 (1978), 372–413.

Euclides, *Elementa*, vol. i, ed. E. S. Stamatis (after J. L. Heiberg), Leipzig, 1969.

—, *Data*, ed. H. Menge, Leipzig, 1896.

Eustratius, *In Analyticorum Posteriorum librum secundum Aristotelis commentarium*, ed. M. Hayduck, Berlin, 1907.

Heath, T. L., *Greek Mathematics*, vol. i, Oxford, 1921.

—, *The Thirteen Books of Euclid's Elements*, vols. i & ii, Cambridge, 1st edition (1908), 2nd edition (1926).

—, *Mathematics in Aristotle*, Oxford, 1949.

Heiberg, J. L., 'Mathematisches zu Aristoteles', *Abhandlungen zur Geschichte der Mathematischen Wissenschaften*, 18 (1904), 1–49.

Knorr, W. R., 'Archimedes and the pre-Euclidean Proportion Theory', *Archives Internationales d'Histoire des Sciences*, 28 (1978), 183–244.

Mugler, Charles, *Dictionnaire historique de la terminologie géométrique des Grecs*, Paris, 1958.

Niphius, Augustinus, *Expositiones in Aristotelis Libros Metaphysices*, Venice, 1559.

[40] This assumes, of course, that Philoponus wrote the commentary on the second book of the *An. Post.* which bears his name. I consider the argumentation of this part of the commentary so atrocious, however, as to doubt that it is by Philoponus at all. The account of the duplication of the cube in the commentary to Book I is superb (cf. pp. 102–105). Although Philoponus certainly cribbed that account from another source, even to have done so displays much more intelligence and learning than I would want to attribute to the bungling author of the commentary on the second book. The commentary on the second book is also somewhat terser than one might expect of Philoponus.

[41] I should like to thank J. O. Urmson, G. E. L. Owen, W. R. Knorr, G. Striker, M. Burnyeat, A. A. Long, H. Gelber, P. Suppes, and the editors, for their helpful criticisms and advice in our discussions of this paper. I delivered it to the Philosophy Department, Florida State University, after which there was useful discussion.

Philoponus, *In Analyticorum Posteriorum Aristotelis Libros*, ed. M. Wallies, Berlin, 1905.
—, *Exegesis in Metaphysica*, cod. Urbinas 49.
Proclus, *In Primum Euclidis Elementorum Librum Commentarii*, ed. G. Friedlein, Leipzig, 1873.
Ross, W. D., *Aristotle's Metaphysics*, 2 vols., Oxford, 1924.
—, *Aristotle's Prior and Posterior Analytics*, Oxford, 1949.
Schwegler, A., *Die Metaphysik des Aristoteles*, vol. iv, Frankfurt am Main, 1846.
Simplicius, *In Aristotelis Physicorum* . . ., vol. i, ed. H. Diels, Berlin, 1882.
Walzer, 'On the Arabic Versions of Books A, α, and Λ of Aristotle's Metaphysics', ch. vi of *Greek into Arabic*, Oxford, 1962.